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A NOTE ON THE PARTIAL SUMS OF $\zeta(s)$

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A note on the partial sums of $\zeta(s)$

by

J. van de Lune

ABSTRACT

Defining $\zeta_N(s) = \sum_{n=1}^{N} n^{-s}$, JESSEN proved that if $N \leqq 5$ then the real part of $\zeta_N(\sigma+it)$ is positive for $\sigma \geqq 1$, $t \in \mathbb{R}$.

Using an electronic computer, SPIRA showed that if $N \leqq 9$ then $\zeta_N(\sigma+it) \neq 0$ for $\sigma \geqq 1$, $t \in \mathbb{R}$.

In this note the author discusses the relation of these matters to a problem in approximation theory and proves (not making use of a computer) that $\mathrm{Re}\ \zeta_6(1+it) > 0$ for all $t \in \mathbb{R}$.

Finally he conjectures that if N is any positive integer then $\zeta_N(1+it) \neq 0$ for all $t \in \mathbb{R}$.

KEY WORDS & PHRASES: *Partial sums (sections) of the zeta-function, zeros, Tauberian theorems, approximation theory.*

We start this note by recalling WIENER's general tauberian theorem for the real line.

For any $f \in L^1(\mathbb{R})$ and any $h \in \mathbb{R}$ let $f_h \in L^1(\mathbb{R})$ be the h-translate of f, i.e.

$$f_h(x) = f(x+h), \qquad (x \in \mathbb{R})$$

and let T_f denote the span of the set of all translates of f, i.e.

$$T_f = \{ \sum_{n=1}^{N} c_n f_{h_n} \mid N \in \mathbb{N};\ c_n \in \mathbb{C};\ h_n \in \mathbb{R}\}.$$

Then WIENER's general tauberian theorem states that T_f is dense in $L^1(\mathbb{R})$ if and only if the Fourier transform $\hat{f}$ of f does not vanish on $\mathbb{R}$, i.e.

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx \neq 0, \qquad \forall t \in \mathbb{R}.$$

In [4] KOREVAAR showed that WIENER's theorem may be translated into the following theorem for $L^1(0,1)$: If $f \in L^1(0,1)$ then the span of the set of functions

$$\{x^{\lambda-1} f(x^\lambda)\}_{\lambda \in \mathbb{R}^+}, \qquad (0<x<1)$$

is dense in $L^1(0,1)$ if and only if

$$(*) \qquad \int_0^1 f(x) \left(\log \frac{1}{x}\right)^{it} dx \neq 0, \qquad \forall t \in \mathbb{R}.$$

In this note we want to discuss the question whether KOREVAAR's theorem applies to the functions $f_N \in L^1(0,1)$ defined by

$$f_N(x) = \frac{1 - x^N}{1 - x}, \qquad (0<x<1)$$

where N is some fixed positive integer.

We first compute the integral

$$\int_0^1 \frac{1-x^N}{1-x}\left(\log\frac{1}{x}\right)^{it}dx = \int_0^1 (1+x+\ldots+x^{N-1})\left(\log\frac{1}{x}\right)^{it}dx =$$

$$= \int_0^\infty (1+e^{-u}+\ldots+e^{-(N-1)u})\, u^{it}\, e^{-u}\, du =$$

$$= \sum_{n=1}^N \int_0^\infty e^{-nu}\, u^{it}\, du =$$

$$= \sum_{n=1}^N \int_0^\infty e^{-v}\left(\frac{v}{n}\right)^{it}\frac{dv}{n} =$$

$$= \Gamma(1+it)\cdot\sum_{n=1}^N \frac{1}{n^{1+it}}\,, \qquad (t\in\mathbb{R})$$

from which it is clear that f_N satisfies condition (*) of KOREVAAR's theorem if and only if

$$(**) \qquad \zeta_N(1+it) \overset{\text{def}}{=} \sum_{n=1}^N \frac{1}{n^{1+it}} \neq 0, \qquad \forall t\in\mathbb{R}\,.$$

Since for all $t \in \mathbb{R}$

$$\zeta_1(1+it) = 1$$

$$|\zeta_2(1+it)| \geqq 1-\frac{1}{2} = \frac{1}{2}$$

$$|\zeta_3(1+it)| \geqq 1-\frac{1}{2}-\frac{1}{3} = \frac{1}{6}$$

it is clear that (**) holds for $N = 1,2,3$. Note that the case $N = 1$ also follows from the wellknown Stone-Weierstrass theorem.

Following JESSEN, we have for $N = 4$ (compare [3])

$$\operatorname{Re} \zeta_4(1+it) = \sum_{n=1}^{4} \frac{1}{n} \cos(t \log n) \geqq$$

(we write $x = t \log 2$)

$$\geqq 1 + \frac{1}{2} \cos x - \frac{1}{3} + \frac{1}{4} \cos 2x =$$

$$= 1 - \frac{1}{3} + \frac{1}{2} \cos x + \frac{1}{4} \{2 \cos^2 x - 1\} =$$

(we write $u = \cos x$)

$$= \frac{5}{12} + \frac{1}{2}(u+u^2) \geqq$$

$$\geqq \frac{5}{12} + \frac{1}{2} \min_{|u| \leqq 1} (u+u^2) = \frac{5}{12} + \frac{1}{2} \cdot (-\frac{1}{4}) = \frac{7}{24}$$

so that

$$\operatorname{Re} \zeta_4(1+it) \geqq \frac{7}{24}, \qquad \forall t \in \mathbb{R}.$$

Consequently we have

$$\zeta_4(1+it) \neq 0, \qquad \forall t \in \mathbb{R}.$$

From the above calculation it is also clear that

$$\operatorname{Re} \zeta_5(1+it) \geqq \operatorname{Re} \zeta_4(1+it) - \frac{1}{5} \geqq \frac{7}{24} - \frac{1}{5} = \frac{11}{120}$$

so that also

$$\zeta_5(1+it) \neq 0, \qquad \forall t \in \mathbb{R}$$

a result which is also due to JESSEN (cf. [3]).

The case $N = 6$ is somewhat less transparant. In [1] SPIRA claims to have shown that

$$(***) \qquad \operatorname{Re} \zeta_N(s) > 0, \qquad (\operatorname{Re} s \geqq 1)$$

for N = 6 and N = 8 and promises to return to these matters in [2].

However, in [2] SPIRA presents *machine proofs*.

In this note we will present a theoretical proof of the following

PROPOSITION. Re $\zeta_6(1+it) > 0, \qquad \forall t \in \mathbb{R}.$

PROOF. In this proof we will write

$$x = t \log 2 \; ; \quad y = t \log 3$$

$$u = \cos x \; ; \quad v = \cos y.$$

We will proof our proposition by showing that Re $\zeta_6(1+it) > \frac{1}{100}$, $\forall t \in \mathbb{R}$.

Observe that

$$\mathrm{Re}\, \zeta_6(1+it) = \sum_{n=1}^{6} \frac{1}{n} \cos(t \log n) =$$

$$= 1 + \frac{1}{2}\cos x + \frac{1}{3}\cos y + \frac{1}{4}\cos 2x + \frac{1}{5}\cos(t \log 5) + \frac{1}{6}\cos(x+y) \geqq$$

$$\geqq 1 + \frac{1}{2}\cos x + \frac{1}{3}\cos y + \frac{1}{4}(2\cos^2 x - 1) - \frac{1}{5} +$$

$$+ \frac{1}{6}(\cos x \cos y - \sin x \sin y) \geqq$$

$$\geqq \frac{11}{20} + \frac{1}{2}(u+u^2) + \frac{1}{3}v + \frac{1}{6}uv - \frac{1}{6}\sqrt{1-u^2}\,\sqrt{1-v^2} =$$

$$\stackrel{\text{def}}{=} \phi(u,v), \qquad (-1 \leqq u\, ,\, v \leqq 1).$$

It is clear that ϕ is continuous on the compact square $-1 \leqq u, v \leqq 1$ so that ϕ assumes an absolute minimum.

The partial derivatives of ϕ on the open square $-1 < u, v < 1$ may be written as

$$\frac{\partial\phi}{\partial u} = \frac{1}{2} + u + \frac{1}{6}v + \frac{u}{6}\frac{\sqrt{1-v^2}}{\sqrt{1-u^2}}$$

and

$$\frac{\partial\phi}{\partial v} = \frac{1}{3} + \frac{1}{6}\, u + \frac{v}{6}\, \frac{\sqrt{1-u^2}}{\sqrt{1-v^2}} .$$

We first prove that ϕ assumes its minimal value in the *interior* of the square $-1 \leqq u , v \leqq 1$.

a. On the segment $v = 1$, $-1 \leqq u \leqq 1$, we have

$$\phi(u,v) = \phi(u,1) = \frac{53}{60} + \frac{2}{3}\, u + \frac{1}{2}\, u^2$$

which is minimal for $u = -\frac{2}{3}$ with minimal value $\frac{119}{180}$. At $(u,v) = (-\frac{2}{3}, 1)$ we have $\frac{\partial\phi}{\partial v} = +\infty$ so that ϕ does *not* assume its minimal value on the segment $v = 1$, $-1 \leqq u \leqq 1$.

b. On the segment $u = 1$, $-1 \leqq v \leqq 1$ we have

$$\phi(u,v) = \phi(1,v) = \frac{31}{20} + \frac{1}{2}\, v$$

which is minimal for $v = -1$ with minimal value $\frac{21}{20}$ $(> \frac{119}{180})$ so that ϕ does *not* assume its minimal value on the segment $u = 1$, $-1 \leqq v \leqq 1$.

c. For $v = -1$, $-1 \leqq u \leqq 1$ we have

$$\phi(u,v) = \phi(u,-1) = \frac{13}{60} + \frac{1}{3}\, u + \frac{1}{2}\, u^2$$

which is minimal for $u = -\frac{1}{3}$ with minimal value $\frac{29}{180}$. Since $\frac{\partial\phi}{\partial v}(-\frac{1}{3}, -1) = -\infty$ it follows that ϕ does *not* assume its minimal value on the segment $v = -1$, $-1 \leqq u \leqq 1$.

d. Finally, for $u = -1$, $-1 \leqq v \leqq 1$, we have

$$\phi(u,v) = \phi(-1,v) = \frac{11}{20} + \frac{1}{6}\, v$$

which is minimal for $v = -1$ with minimal value $\frac{23}{60}$ $(> \frac{29}{180})$ so that ϕ does *not* assume its minimal value on the segment $u = -1$, $-1 \leqq v \leqq 1$.

It follows that we may restrict ourselves to the open square $-1 < u , v < 1$.

In order to show that the minimal value of ϕ is positive we consider the equations

$$\frac{\partial\phi}{\partial u} = \frac{\partial\phi}{\partial v} = 0, \qquad (-1<u\,,\ v<1).$$

Observe that if $u \geqq 0$ then

$$\frac{\partial\phi}{\partial u} \geqq \frac{1}{2} + \frac{1}{6}v > \frac{1}{2} - \frac{1}{6} > 0$$

and that if $v \geqq 0$ then

$$\frac{\partial\phi}{\partial v} \geqq \frac{1}{3} + \frac{1}{6}u > \frac{1}{3} - \frac{1}{6} > 0$$

so that we only need to consider $\phi(u,v)$ on the open square $-1 < u\,,v < 0$.

The equations $\frac{\partial\phi}{\partial u} = 0$ and $\frac{\partial\phi}{\partial v} = 0$ are equivalent to

$$3 + 6u + v + u\frac{\sqrt{1-v^2}}{\sqrt{1-u^2}} = 0$$

respectively

$$2 + u \qquad + v\frac{\sqrt{1-u^2}}{\sqrt{1-v^2}} = 0,$$

from which we obtain

$$\frac{3 + 6u + v}{u} = \frac{v}{2 + u}$$

or, equivalently, that

$$v = -\frac{1}{2}(6+15u+6u^2).$$

Since $6 + 15u + 6u^2 = 0$ for $u = -\frac{1}{2}$ and $u = -2$ and $-\frac{1}{2}(6+15u+6u^2) = -1$ for

$$u = \frac{-15 + \sqrt{129}}{12} \qquad \text{and} \qquad u = \frac{-15 - \sqrt{129}}{12}\ (< -1)$$

we find that ϕ is minimal on the curve

$$v = -\frac{1}{2}(6+15u+6u^2)$$

where

$$-\frac{1}{2} < u < \frac{-15+\sqrt{129}}{12} \quad (< -\frac{1}{4}).$$

From $\frac{\partial\phi}{\partial v} = 0$ it follows that

$$\frac{1}{3} + \frac{1}{6}u = \frac{-v}{6}\frac{\sqrt{1-u^2}}{\sqrt{1-v^2}}$$

from which we obtain subsequently

$$\frac{1}{3} + \frac{1}{6}(-\frac{1}{2}) < \frac{-v}{6\sqrt{1-v^2}},$$

$$\frac{3}{2} < \frac{-v}{\sqrt{1-v^2}},$$

$$\frac{9}{4} < \frac{v^2}{1-v^2} = -1 + \frac{1}{1-v^2},$$

$$v^2 > \frac{9}{13},$$

$$-v = |v| > \frac{3}{\sqrt{13}} \quad (> \frac{3}{4}).$$

Hence, if u and v satisfy the restrictions described above we have

$$\phi(u,v) > \frac{11}{20} + \frac{1}{2}\inf(u+u^2) - \frac{1}{3} + \frac{1}{6}(\inf|u|)(\inf|v|) - \frac{1}{6}\sqrt{1-\frac{1}{16}}\sqrt{1-\frac{9}{13}} >$$

$$> \frac{11}{20} - \frac{1}{8} - \frac{1}{3} + \frac{1}{6}\cdot\frac{1}{4}\cdot\frac{3}{4} - \frac{1}{12}\frac{\sqrt{15}}{\sqrt{13}} >$$

$$> \frac{11}{20} - \frac{1}{8} - \frac{1}{3} + \frac{1}{32} - \frac{1}{12}\cdot\frac{4}{3} =$$

$$= \frac{11}{20} + \frac{1}{32} - \frac{1}{8} - \frac{4}{9} >$$

$$> 0.55 + 0.03 - 0.125 - 0.445 = 0.01,$$

so that Re $\zeta_6(1+it) > 0.01$ for all $t \in \mathbb{R}$, proving the proposition. $\square$

REMARK. Numerical computations show that ϕ assumes its minimal value 0.1197... at the point $u = -0.3266\ldots$, $v = -0.8705\ldots$.

Computing Re $\zeta_7(1+it)$ for $t = n.10^{-1}$, $(n=1,2,3,\ldots)$, we found that

$$\operatorname{Re} \zeta_7(1+it) = -0.0136\ldots < 0 \quad \text{for } t = 1{,}009$$

so that the above proposition does not hold when ζ_6 is replaced ζ_7.

We conclude this note by stating the following

CONJECTURE. For every positive integer N one has that

$$\zeta_N(1+it) \neq 0, \qquad \forall t \in \mathbb{R}.$$

REFERENCES

[1] R. SPIRA, *Zeros of sections of the zeta-function, I*, Math. Comp., vol. 20, (1966) pp.542-550.

[2] R. SPIRA, *Zeros of sections of the zeta-function, II*, Math. Comp., vol. 22, (1968) pp.163-173.

[3] P. TURÁN, *On some approximative Dirichlet polynomials in the theory of the zeta-function of Riemann*, Danske Vid. Selsk. Math. Fys. Medd., vol. 24, 17 (1948) pp.3-36.

[4] WISKUNDIGE OPGAVEN, Noordhoff N.V., Groningen (1943) pp.419-425.